AN INTRODUCTION TO 3-MANIFOLDS

by

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Preface

These notes cover a series of six lectures which I gave at the University of Maryland in December, 1973. My aim in the course was to head straight into the subject and prove a few theorems, not to give a complete survey of 3-manifold theory. However, I started with an introductory lecture to set the scene in the problem of classifying 3-manifolds.

The main theorem of the course is Waldhausen's important result that homotopy equivalent, closed, sufficiently large 3-manifolds are homeomorphic and his various extensions. I do not give Waldhausen's proof but present his results as following on from a proof of Kneser's Conjecture and its extensions to the case when the surface is not a sphere. These results demonstrate the power of some of the methods used today in 3-manifold theory. The course ends with a proof of the coherence of fundamental groups of 3-manifolds and some examples of 'wild' group action on R^3.

In order not to obscure the simplicity of the methods used, and of course to save time, I restricted myself to the orientable case in much of the proofs. Analogous results do hold in general and the proofs follow the same plan, but they need more work at some points. A far more serious gap for a course claiming to be an introduction to 3-manifolds is the omission of proofs of the Loop Theorem and Sphere Theorem. This had to be so for obvious reasons of time.

All the results and proofs here have appeared in print before.

Chapter I. Introductory Lecture

The basic problem of manifold theory is that of classification. I want to start by discussing why dimension 3 has a special position in this problem. Of course, there are really three classification problems, one for each of the categories of smooth, piecewise-linear and topological manifolds. We call these categoreis DIFF, PL and TOP respectively.

In the case of dimension 5 or more, we now know from the work of Kirby and Siebenmann [11], that the PL and TOP classifications of manifolds are different. In the case of dimension 7 or more, we know that the PL and DIFF classifications are different [10,14]. However, in dimension 3 or less, all three classifications are the same. The original proof of this fact for PL and TOP in dimension 3 was by Moise [17]. It can also be proved using the ideas of Kirby and Siebenmann. It will usually be convenient to use the PL category in proofs. Further the classification of manifolds of dimensions 1 or 2 is complete even in the non-compact case [20,22]. The fact that we know all about submanifolds of 3-manifolds which are of codimension one plays a key role in the theory.

One other important special property of dimension 3 is the following. Any countable group is the fundamental group of a 4-manifold and any finitely presented group is the fundamental group of a closed 4-manifold. Neither of these statements is true in dimension 3. This implies that the classification problem in dimension 4 or more is not solvable even if we restrict our attention to compact manifolds because the classification of finitely presented groupsis not possible [13]. By this one means that there can be no algorithm for classifying manifolds. It is not known whether or not the classification problem in dimension three is solvable.

For the rest of this lecture, I want to set the scene for looking at the classification problem in dimension 3. I will consider mainly compact, orientable 3-manifolds.

Two good classes of examples to keep in mind are lens spaces and knot complements. The lens space $L(m,n)$, where m and n are coprime, is the quotient of the 3-sphere S^3 by a free Z_m-action. This action is defined by taking $S^3 = \{(z_1,z_2) : z_i$ is a complex number and $|z_1|^2 + |z_2|^2 = 1\}$ and mapping (z_1,z_2) to $(e^{2\pi i/m} z_1, e^{2\pi i/m} z_2)$. The fundamental group of $L(m,n)$ is Z_m. By a knot complement, I mean S^3 with the interior of a closed regular neighborhood of a knot removed. Thus the boundary of a knot complement is a torus $S^1 \times S^1$.

One can construct more examples of 3-manifolds in various ways. First, if G is a subgroup of $\pi_1(M^3)$, then M has a covering space X with $\pi_1(X) = G$. But as X is a covering space of M, X must also be a 3-manifold. Another method of construction is to take the connected sum of two 3-manifolds M_1 and M_2. This we denote by $M_1 \# M_2$. It is defined by choosing two 3-balls B_1 and B_2 in the interiors of M_1 and M_2. We then remove the interior of B_i from M_i and identify the two boundary spheres by a homeomorphism to obtain $M_1 \# M_2$. Note that, strictly speaking, $M_1 \# M_2$ may not be well-defined as there are two non-isotopic homeomorphisms of S^2. Van Kampen's Theorem tells us that $\pi_1(M_1 \# M_2) \cong \pi_1(M_1) * \pi_1(M_2)$, because $\pi_1(M_i) = \pi_1(M_i - B_i)$ as M_i is 3-dimensional.

Yet another particularly interesting construction is to take two knot complements and glue their boundaries together by a homeomorphism. If we choose complements K_1 and K_2 of non-trivial knots, then

$\pi_1(K_1 \cup K_2) \cong \pi_1(K_1) *_{Z\times Z} \pi_1(K_2)$, because $\pi_1(\partial K_i)$ injects into $\pi_1(K_i)$. Now for any two knot complements K_1 and K_2 , there is a homeomorphism $h : \partial K_1 \to \partial K_2$ such that $K_1 \cup_h K_2$ is a homology 3-sphere. Thus we have a huge supply of homology spheres with infinite fundamental group.

Let us return to the connected sum operation for a time. We observe that for any 3-manifold M , $M \# S^3$ is homeomorphic to M . Thus S^3 is a unit in the abelian monoid formed by homeomorphism classes of 3-manifolds. We want to consider the question of factorising in this monoid. We say a 3-manifold M is <u>prime</u> if whenever $M = M_1 \# M_2$ then one of M_1 or M_2 is S^3 . Alexander proved that S^3 is prime [1]. We can now state the following result proved by Kneser [12] and Milnor [15].

<u>Theorem 1.1</u> A compact, orientable 3-manifold M can be factored uniquely (up to order of factors) as a finite connected sum of prime 3-manifolds.

It is convenient to make a further definition. We say M^3 is <u>irreducible</u> if whenever we embed S^2 in M then S^2 bounds a 3-ball D^3 in M .

<u>Lemma 1.2</u> If M is a compact, orientable, prime 3-manifold, then $M = S^1 \times S^2$ or M is irreducible.

Proof: As M is prime, every S^2 embedded in M which separates M bounds a 3-ball. Suppose M admits a non-separating 2-sphere S , which we can suppose embedded in the interior of M . Choose a regular neighborhood $V = S^2 x[-1,1]$ of S and an embedded path Γ in $M - V$ from a point of $S^2 x\{-1\}$ to a point of $S^2 x\{1\}$. The sphere S'

which is the connected sum of $S^2 x\{-1\}$ with $S^2 x\{1\}$ along the boundary of a regular neighborhood U of Γ clearly bounds $U \cup V$ in M . It follows that $M = U \cup V \cup D^3$ and M must be homeomorphic to $S^1 \times S^2$. In general, M might be the non-trivial S^2 bundle over S^1 ,but we know M to be orientable.

Thus the classification of compact orientable 3-manifolds reduces to the classification of compact orientable irreducible 3-manifolds. We now consider such a manifold M . There are two cases according to whether $\pi_1(M)$ is finite or not. We let $\tilde{M}$ denote the universal covering space of M .

<u>Case 1</u> $\pi_1(M)$ <u>is finite</u>

If M is not closed, we consider the exact sequence $H_2(M, \partial M) \to H_1(\partial M) \to H_1(M)$. We know $H_1(M)$ is finite and $H_2(M, \partial M) \cong H^1(M) = 0$. Hence $H_1(\partial M)$ is finite. If follows that ∂M is a union of 2-spheres and so $M = D^3$. (Note the use of the classification of surfaces here).

If M is closed, then $\tilde{M}$ is also closed. Also $\pi_1(\tilde{M})$ is trivial and $\pi_2(\tilde{M}) \cong H_2(\tilde{M}) \cong H^1(\tilde{M}) = 0$. Hence $\tilde{M}$ is a homotopy 3-sphere.

Thus in the case of finite fundamental group, the classification problem reduces to solving the Poincaré Conjecture and finding out about free group actions on homotopy spheres. Milnor [16] has listed the groups which can act freely on homotopy 3-spheres.

We observe that even neglecting the Poincaré Conjecture the classification may well be very complicated. For there exist lens spaces which have isomorphic fundamental groups yet are not homotopy equivalent [37].

Also there exist homotopy equivalent lens spaces which are not homeomorphic [21,37].

We now turn to the other possibility. At this point we must quote the Sphere Theorem of Papakyriakopoulos [18] and Whitehead [36].

<u>Sphere Theorem</u> If M^3 is orientable and $\pi_2(M)$ is non-zero, then there exists an embedding of S^2 in M which represents a non-zero element of $\pi_2(M)$.

It follows that if M is irreducible, then $\pi_2(M) = 0$.

<u>Case 2</u> $\pi_1(M)$ <u>is infinite</u>

$\tilde{M}$ is not compact and so $H_3(\tilde{M}) = 0$. Also $H_1(\tilde{M}) = 0$ and $H_2(\tilde{M}) = \pi_2(\tilde{M}) = \pi_2(M) = 0$. Hence $H_i(\tilde{M}) = 0$ for all $i > 0$ and Whitehead's theorem tells us that $\tilde{M}$ is contractible. Therefore M is aspherical. ie. $\pi_i(M) = 0$ for $i > 1$. This gives us new information about $\pi_1(M)$. In particular, $\pi_1(M)$ has cohomological dimension at most 3. Also $\pi_1(M)$ must be torsion free, for if Z_n were a subgroup of $\pi_1(M)$, there would be a covering space X of M with $\pi_1(X) = Z_n$. The space X would be aspherical and 3-dimensional, which is a contradiction, as the homology of Z_n is periodic with period 2.

We say that M is a $K(\pi,1)$. i.e. an aspherical space whose fundamental group is π . The fact that any irreducible orientable 3-manifold with infinite fundamental group is a $K(\pi,1)$ goes a long way toward explaining the very special role that the fundamental group plays in dimension 3.

We close this discussion by observing that in all known examples $\tilde{M}$ is R^3 if M is closed, and otherwise $\tilde{M}$ is D^3 with some of ∂D^3 removed.

In some ways, the classification problem in this case seems more tractable than in the first case. For example, Waldhausen's results (Chapter IV) suggest that homotopy equivalent closed orientable irreducible 3-manifolds with infinite fundamental group must be homeomorphic. As these manifolds are $K(\pi,1)$'s, they are homotopy equivalent whenever their fundamental groups are isomorphic, thus they are determined by their fundamental group alone. However, in contrast to the first case, we are very far from characterizing those groups which can occur as the fundamental groups of such manifolds. If we restrict the groups to be abelian, nilpotent or solvable there is a complete classification [2,5,32].

There are a collection of results, which I call structure theorems, which have a mainly group-theoretic hypothesis and a geometric conclusion. We give some examples below.

Kneser's Conjecture, which is proved in Chapter III has the hypothesis that M^3 is closed with $\pi_1(M) = G_1 * G_2$ and the conclusion is that M is $M_1 \# M_2$ with $\pi_1(M_i) = G_i$. This sort of result gives information about the manifold M and hence gives new information about $\pi_1(M)$. Thus we obtain some information, in rather special cases, about the structure of $\pi_1(M)$.

If M^3 is a bundle over S^1 with fibre a surface then $\pi_1(M)$ is an extension of a surface group by Z. Stallings' fibration theorem [26] is a converse to this—the key hypothesis being only that $\pi_1(M)$ is an extension of some finitely generated group G by Z. Thus in this theorem G turns out to be a surface group.

If M_1 and M_2 have boundary components F_1 and F_2 such that $F_1 \cong F_2$ and the natural maps $\pi_1(F_i) \to \pi_1(M_i)$ are injective and if M is obtained by gluing F_1 to F_2 then $\pi_1(M) = \pi_1(M_1) *_{\pi_1(F_1)} \pi_1(M_2)$. In Chapter III we prove a converse of this. This is a generalization of Kneser's conjecture. If $M \cong F^2 \times S^1$ for some surface F then $\pi_1(M) = \pi_1(F) \times Z$. Epstein [3] has proved a converse of this where the group-theoretic hypothesis is only that $\pi_1(M) = A \times B$ and is infinite.

Chapter II. Group Theory — theorems of Kuros and Grusko

As we pointed out in the previous chapter, the fundamental group plays a very important rôle in 3-manifold theory. The sort of group theory which is needed is that part of the subject to do with free groups, amalgamated free products and cohomology of groups. I want to present here proofs of Grusko's theorem and of the Kuros subgroup theorem both with a topological audience in mind. I do this partly for completeness and partly because the methods of proof will be used on many other occasions.

The basic idea in these topological proofs of group theoretic results is that given a presentation of a group G, there is a corresponding 2-dimensional CW-complex X with $\pi_1(X) \cong G$. We give X one 0-cell, and the 1-cells of X correspond to the generators of G, the 2-cells of X correspond to the relations of G. Of course, one needs van Kampen's theorem to do this which is clearly irrelevant to a group theoretic result. One can set up these proofs more abstractly in terms of groupoids to get around this [9].

Whenever possible, I will suppress base points. The reader can insert his own!

Kuros subgroup theorem

If H is a subgroup of $G = A * B$, then H is the free product of a free group with subgroups of conjugates of A or B.

Corollary 2.1 If H is indecomposable, ie. H is not a free product, then H lies in a conjugate of A or B or $H \cong Z$.

Proof of the subgroup theorem

Let X be a simplicial complex with $\pi_1(X,e) = A$ where e is a vertex of X . Let Y be a simplicial complex with $\pi_1(Y,e') = B$ where e' is a vertex of Y . Let K be the union of X and Y and a 1-simplex E where the ends of the 1-simplex are identified with e and e' . Then $\pi_1(K,e) = A * B$. If $H \subset G$, then $H = \pi_1(L)$ for some covering space L of K with projection map $p : L \to K$. Now inside L we have $p^{-1}(X)$ which is a covering space of X and so consists of various connected covering spaces of X . Similarly for $p^{-1}(Y)$. Finally, as E is simply connected, $p^{-1}(E)$ is a union of copies of E . Thus L looks like a graph with covering spaces of X or Y at each vertex. The base point of L is a point * such that $p(*) = e$, and the result is now clear. We get a free group coming in if the graph is not a tree.

We now proceed to a proof of Grusko's theorem which is a much more difficult result, needing a more subtle proof. The proof we give is due to Stallings [27].

Grusko's Theorem Let F be a finitely generated free group, $G = G_1 * G_2$, and let $f_*: F \to G$ be an epimorphism. Then there are subgroups F_1 and F_2 of F such that $F = F_1 * F_2$ and $f_*(F_i) = G_i$.

Corollary 2.2 If $\mu(G)$ is the minimal number of generators of G, then $\mu(G) = \mu(G_1) + \mu(G_2)$. In particular each G_i is finitely generated.

Proof: This corollary follows because $\mu(F) = \mu(F_1) + \mu(F_2)$, which can be proved by abelianizing.

Corollary 2.3 Any finitely generated group G is a finite free product of indecomposable groups. The factors in such a decomposition are unique up to isomorphism and order of the factors. Further those factors not isomorphic to Z are unique up to conjugacy in G.

Proof: The existence of such a decomposition follows from Corollary 2.2. The uniqueness results all follow from the Kuros subgroup theorem and Corollary 2.1.

Proof of Grusko's Theorem

There is a 1-complex K with $\pi_1(K) = F$ and a 2-complex L with $\pi_1(L) = G$ and a simplicial map $f\colon K \to L$, such that $f_*\colon \pi_1(K) \to \pi_1(L)$ is the given map f_*. We can choose L to be the union of L_1, L_2 and a 1-simplex E, as in the proof of the Kuros subgroup theorem. Thus $\pi_1(L_i) = G_i$.

Let v be the midpoint of E and let us change notation slightly so that L_i denotes the closure of a component of $L - v$. Note that $f^{-1}(v)$ cannot be empty unless G_1 or G_2 is trivial in which case the result is obvious. Otherwise our aim is to obtain $f^{-1}(v)$ equals a tree T in K. In this situation we would have T separating K into two components K_1 and K_2 with $f(K_i) \subset L_i$.

Let $F_i = \pi_1(K_i)$. Then $\pi_1(K) = \pi_1(K_1) *_{\pi_1(T)} \pi_1(K_2) = F_1 * F_2$ and $f_*(F_i) = G_i$ which is the required result.

We now consider the general situation. By a homotopy of f , we can suppose that $f^{-1}(v)$ consists of finitely many points because K is 1-dimensional. (Observe that this is the only point where the hypothesis that F is free enters the proof.) Assuming that $f^{-1}(v)$ is not connected, we are going to describe a method for replacing the given situation by another in which the new $f^{-1}(v)$ has less components. These components will be trees and not necessarily points anymore. So what we really have to do is to show how to reduce the number of components of $f^{-1}(v)$ when each component is a tree. Once we can do this we can repeat until $f^{-1}(v)$ is a tree in K and the result will follow.

We now come to Stallings' method of arc-chasing, which we apply to the situation where $f^{-1}(v)$ is a disjoint union of at least two trees. Choose two distinct components A and B of $f^{-1}(v)$. Choose a path Γ' in K from some point e of A to some point of B . By a path we mean simply a map $I \to K$, and not necessarily an embedding. Then $f(\Gamma')$ is a loop in L based at v . As $f_*: F \to G$ is onto, there is a loop γ in K based at e with $f(\gamma)$ homotopic to $f(\Gamma')$. We consider the path $\Gamma = \Gamma'\gamma^{-1}$ from e in A to some point of B .

Clearly $f(\Gamma)$ is a loop in L based at v which is null-homotopic. We can suppose that Γ is simplicial. Thus we can express Γ as a union of subpaths $\Gamma_1,\ldots,\Gamma_n$ such that the ends of each Γ_i lie in $f^{-1}(v)$ and $f(\Gamma_i)$ lies in L_1 or L_2 . Further we can suppose that the $f(\Gamma_i)$'s alternate between L_1 and L_2 . Note that this means that Γ_i may meet $f^{-1}(v)$ in its interior but this does not matter.

Let g_i denote the homotopy class of the loop $f(\Gamma_i)$ in $\pi_1(L,v) = G$. As $f(\Gamma)$ is null-homotopic, we have the equation $g_1 g_2 \cdots g_n = 1$ in $G = G_1 * G_2$, where the g_i's lie alternately in G_1 and G_2 . This implies that some g_i equals 1. Let the components of $f^{-1}(v)$ in which the two end points of Γ_i lie be P and Q .

If $P = Q$, then we can change Γ to Γ'' by removing Γ_i and inserting an appropriate path in P . Clearly this new path Γ'' joins the distinct components A and B of $f^{-1}(v)$ and $f(\Gamma'')$ is a contractible loop in L . Also Γ'' is a union of only $(n-1)$ subpaths $\Gamma_1,\ldots,\Gamma_{i-1},\Gamma_{i+1},\ldots,\Gamma_n$. Thus if we repeat the above construction enough times, we must arrive at a subpath Γ_i of Γ which joins distinct components P and Q of $f^{-1}(v)$, and which has $f(\Gamma_i)$ as a contractible loop in L_1 or L_2 . This is what Stallings calls a binding tie.

We now replace K by a new complex K_1 described as follows. Let D denote a 2-disc and divide its boundary into two arcs α and β with common end points. We can define a map $f_1\colon D \to L$ such that $f_1(\alpha) = f(\Gamma_i)$, $f_1(\beta) = v$ and $f_1^{-1}(v) \subset \partial D$, because $f(\Gamma_i)$ is a null-homotopic loop in L_1 or L_2 . Let K_1 be the union of K and D with α identified to Γ_i in K and define $f_1\colon K_1 \to L$ by setting f_1 equal to f on K . Clearly K_1 deformation retracts to K . Also $f_1^{-1}(v) = f^{-1}(v) \cup \beta$, which is a union of trees and has one less component than $f^{-1}(v)$.

This completes the reduction step and hence the proof of the theorem.

We assume the reader is familiar with the amalgamated free product construction of groups written $A *_C B$. This is the fundamental group of a suitable union of two spaces with connected intersection. We will also want to use the Higman-Neumann-Neumann (HNN) construction, which comes from one group A and two embeddings i_0 and i_1 of a group C into A . The resulting group, written $A*_C$, is the fundamental group of the union of a $K(A,1)$ with $K(C,1) \times I$, where we map $K(C,1) \times \{0,1\}$ to $K(A,1)$ corresponding to i_0 and i_1 . The group $A *_C$ has presentation $\{A, s : s^{-1} i_0(c) s = i_1(c), \text{for all } c \in C\}$. It is a fact that A is naturally embedded in $A *_C$, but clearly A does not generate it.

Chapter III Splitting Theorems

The Loop Theorem and its extensions play an essential role in all the theorems we consider here. We will state the Loop Theorem without proof and sketch proofs for the extensions we need.

We say a map $f : M \to N$ of manifolds is <u>proper</u> if $f^{-1}(\partial N) = \partial M$. We say that a connected surface F embedded in a 3-manifold M is <u>incompressible</u> if the natural map $\pi_1(F) \to \pi_1(M)$ is injective. If F is not connected, we say that F is <u>incompressible</u> if every component of F is. Otherwise F is <u>compressible</u>.

<u>Loop Theorem</u> (Papakyriakopoulos [19], Stallings [28])

If F is a boundary component of M^3 and is compressible in M, then there is a properly embedded 2-disc D^2 in M with $\partial D^2 \subset F$ and ∂D^2 is essential in F.

The following extensions of the Loop Theorem are due to Stallings [26].

<u>Corollary 3.1</u> If F is a compact surface, properly and 2-sidedly embedded in M^3, which is compressible, then there is a 2-disc D^2 embedded in the interior of M with $D^2 \cap F = \partial D^2$ essential in F.

Proof: First consider the case when F is connected. Remove all the boundary of M, then cut M along F, to get M' with two copies of F as its boundary. One of these copies must be compressible in M' by van Kampen's theorem and facts about amalgamated free products. The result follows.

If F is not connected, we can apply the above to a compressible component F_λ of F to obtain $D^2 \subset M$ with $\partial D^2 \subset F_\lambda$ but D^2 may meet other components of F . Put D^2 transverse to $F - F_\lambda$, and consider an innermost circle C of $D^2 \cap F$. Of course, C bounds a sub-disc E of D . If C is essential in F , then the 2-disc E is the required 2-disc. Otherwise C bounds a 2-disc E' in F and we can replace D by $D - E + E'$ to reduce the number of circles in $D \cap F$. By repeating this process, we must arrive at a 2-disc which meets F only in its boundary and whose boundary is essential in F as required.

Corollary 3.2 Suppose that M^3 is compact, K is a simplicial complex with a principal 1-simplex with midpoint v and $\pi_2(K) = 0$. Then given a map $f : M \to K$ there is a PL map $g: M \to K$ homotopic to f which is transverse to v and such that $g^{-1}(v)$ is incompressible in M .

Remark As g is transverse to f , $g^{-1}(v)$ is compact, and 2-sided and properly embedded in M .

Proof of Corollary 3.2. First we can homotop f to be PL and transverse to v . Note that this is a very trivial transversality result. In fact, one has only to choose a point v' in the interior of the principal 1-simplex which is not the image of a vertex of M , when M is triangulated so that f is PL . One can arrange $v' = v$ but this is unimportant.

Now consider $f^{-1}(v)$. If $f^{-1}(v)$ is incompressible we are done. Otherwise Corollary 3.1 gives us D^2 lying in the interior of M with $D^2 \cap f^{-1}(v) = \partial D^2$ a circle essential in $f^{-1}(v)$. As $\pi_2(K) = 0$, we can homotop D^2 modulo ∂D^2 so that D^2 maps entirely to v . One can now describe a homotopy of f , fixed outside a regular neighborhood of D , to a map g with $g^{-1}(v)$ equal to $f^{-1}(v)$ surgered upon by D. ie. $g^{-1}(v)$ is obtained from $f^{-1}(v)$ by removing a regular neighborhood of ∂D^2 in $f^{-1}(v)$ and adding two 2-discs parallel to D. We say the homotopy <u>induces surgery by D</u>. This process of surgery reduces the number $\sum_i (2-\chi(F_i))^2$, where the summation is over all components of $f^{-1}(v)$. As $f^{-1}(v)$ is compact, this number is finite and so the process must terminate. At this time, we will have the required map g.

We now come to the first splitting theorem which was proved by Stallings [29]. We give his proof.

<u>Kneser's Conjecture</u>

If M^3 is closed and $\pi_1(M) = A * B$, then $M = M_1 \# M_2$ with $\pi_1(M_1) = A$ and $\pi_1(M_2) = B$.

Proof: We may assume A and B are non-trivial or the result is obvious— and uninteresting.

As before, let K be a $K(A * B,1)$ built out of a $K(A,1)$, $K(B,1)$ and a 1-simplex with midpoint v. As $\pi_2(K) = 0$, there is a map $f : M \to K$ inducing an isomorphism of fundamental groups. By Corollary 3.2, we can homotop f so that f is transverse to v and $f^{-1}(v)$ is incompressible in M. Let L be a component of $f^{-1}(v)$. L must be a closed surface. The commutative diagram

$$\begin{array}{ccc} \pi_1(L) & \to & \pi_1(M) \\ \downarrow & & \downarrow \\ \{1\} = \pi_1(v) & \to & \pi_1(K) \end{array}$$

consists of injections, so we deduce that $\pi_1(L)$ is trivial and L is a 2-sphere.

If $f^{-1}(v)$ is just one 2-sphere, the result is clear. Otherwise, we will show how to homotop f so as to reduce the number of components of $f^{-1}(v)$ by one. Eventually one will arrive at a map g with $g^{-1}(v)$ equal to one 2-sphere. Observe that $f^{-1}(v)$ cannot be empty as A and B are non-trivial.

Now suppose $f^{-1}(v)$ has at least two components. We proceed by the method of arc-chasing again. First choose a path Γ' joining two distinct components of $f^{-1}(v)$. As before, we can arrange that $f(\Gamma')$ is a null-homotopic loop in K based at v . As we are now in a 3-manifold, we can further make Γ' embedded and transverse to $f^{-1}(v)$. Now arguing as before, we obtain a subpath Γ of Γ' joining two distinct components S_1, S_2 of $f^{-1}(v)$ and meeting no other components of $f^{-1}(v)$. Further Γ will meet S_1 and S_2 only in its endpoints and $f(\Gamma)$ is a loop based at v which is null-homotopic. Now, in a similar way to the proof of Corollary 3.2, we can homotop f so as to induce surgery on $f^{-1}(v)$ by Γ . Thus S_1 and S_2 are replaced by $S_1 \# S_2$ where the connected sum is obtained by using the boundary of a regular neighborhood of Γ . This reduces the number of components of $f^{-1}(v)$ by one as required.

We now want to generalize this result to the situation where a closed manifold M is obtained by gluing together two surfaces, not necessarily spheres. We will need to assume that the surfaces are incompressible. Thus $\pi_1(M)$ is of the form $A *_C B$ or $A *_C$ where C is the fundamental group of a surface. We will keep to the orientable case now as this substantially reduces the amount of work needed. For the non-orientable case see [23]. We need to quote the following classical result due to Baer and Nielsen. This result is the 2-dimensional analogue of Waldhausen's result and is most easily proved by the analogous proof to the one we present here for Waldhausen's theorem. Of course, this 2-dimensional analogue is much easier to do.

Theorem 3.3

Suppose M^2, N^2 are compact and orientable and M is not S^2 or D^2. If $f: M \to N$ is a proper map injecting fundamental groups then f is properly homotopic to a map g such that

a) g is a covering map,

or b) $g(M) \subset \partial N$. In this case M must be $S^1 \times I$.

We also need the following result, which though trivial appears not to be well known.

Lemma 3.4 Let F^{n-1} be a closed orientable manifold in the interior of the orientable manifold M^n. Denote by $[F]$ a generator of $H_{n-1}(F,Z) \cong Z$. Then if $[F]$ is non-zero in $H_{n-1}(M,Z)$, it is also indivisible. i.e. the equation $[F] = r\alpha$, $\alpha \in H_{n-1}(M)$, implies r is 1 or -1.

Proof: If F fails to separate M, we can find a circle in M cutting F transversely in one point. This circle represents an element of $H_1(M)$ whose intersection number with $[F]$ is 1 or -1. The result follows as intersection numbers define a homomorphism.

If F separates M into X and Y, then neither X nor Y can be compact with boundary F, or we would have $[F]$ being zero in $H_{n-1}(M)$. Therefore, we can find a path cutting F transversely in one point and ending either in some boundary component of M or 'at infinity'. The same argument now applies using if necessary homology with infinite chains and cohomology with finite chains.

We can now prove the following result, first proved by Feustel [6] and Swarup [30]. The proof we give here seems easier then their proofs.

<u>Lemma 3.5</u> Let M^3 be irreducible and orientable and let F^2 be a closed incompressible surface in M which is not S^2. If $\pi_1(F) \subset G \subset \pi_1(M)$, where G is isomorphic to the fundamental group of a closed orientable surface L, then $G = \pi_1(F)$.

<u>Remark</u>: This result is false in the non-orientable case. For example take $F = \partial M$ where M is a non-trivial I-bundle over L. However this is essentially the only counter example.

<u>Proof of Lemma 3.5</u> Let N be the covering space of M determined by $G \subset \pi_1(M)$. Our embedding of F in M lifts to N. As $\pi_1(F)$ is infinite, so is $\pi_1(M)$, and M is aspherical. Therefore N is also aspherical and so is homotopy equivalent to L. So we have $F \xrightarrow{i} N \xrightarrow{f} L$ where f is a homotopy equivalence. Now $f \circ i$ is homotopic to a covering map of some finite degree r, by Theorem 3.3. Hence $i_*: Z \cong H_2(F) \to H_2(N) \cong H_2(L) \cong Z$ is multiplication by r. Lemma 3.4 now tells us that r is 1 and hence $G = \pi_1(F)$ as required.

This result is what we need to prove the following splitting theorem using the same ideas as in Stallings proof of Kneser's Conjecture. This result has also been proved by Feustel [7].

Theorem 3.6 Let M^3 be closed, orientable and irreducible and suppose $\pi_1(M) = A *_C B$, where C is not equal to A or B and C is isomorphic to the fundamental group of a closed orientable surface F . Then there is an incompressible embedding of F in M separating M into M_1 and M_2 with $\pi_1(M_1) = A$, $\pi_1(M_2) = B$, and $\pi_1(F) = C$.

Remarks F cannot be a 2-sphere. For Kneser's Conjecture would then show that M is not irreducible. The hypothesis that L be orientable can be removed. In fact, this result is true in the non-orientable case if we suppose M is P^2-irreducible. Also the analogous result is true when L fails to separate M . The hypothesis then is that $\pi_1(M) = A *_C$. Neither of the hypotheses that M be irreducible and C be isomorphic to $\pi_1(F)$ can be removed.

Proof of Theorem Let K be a $K(A *_C B, 1)$ constructed by taking $K(A,1)$, $K(B,1)$ and $F \times I$ and mapping $F \times \{0\}$ into $K(A,1)$, $F \times \{1\}$ into $K(B,1)$ suitably. (Note that the resulting space K is aspherical precisely because C injects into A and B . To see this, consider the universal covering space of K , which looks rather like a graph, with a contractible space at each vertex.) As $\pi_2(K) = 0$, there is a map $f: M \to K$ inducing an isomorphism of fundamental groups.

We can suppose by the methods of Corollary 3.2 that f is transverse to $F \times \{1/2\}$ and $f^{-1}(F \times \{1/2\})$ is incompressible in M . Observe that $f^{-1}(F \times \{1/2\})$ cannot be empty now or later as $C \neq A$ and $C \neq B$.

Suppose $f^{-1}(F \times \{1/2\})$ has a component S which is a 2-sphere. Then S bounds a 3-ball in M . As $\pi_3(K) = 0$, we can homotop f so as to remove S , and possibly some other spheres, from $f^{-1}(F \times \{1/2\})$.

Therefore we can suppose that no component of $f^{-1}(F \times \{1/2\})$ is a 2-sphere. Let L be a component. We have $\pi_1(L) \subset f_*^{-1}(\pi_1(F)) \subset \pi_1(M)$ and we deduce from Lemma 3.5 that $\pi_1(L) = f_*^{-1}(\pi_1(F))$. It follows that we can homotop f so that its restriction to L is a homeomorphism onto F .

If $f^{-1}(F \times \{1/2\})$ consists of one copy of F , the result follows. Otherwise we will show how to homotop f so as to reduce the number of components of $f^{-1}(F \times \{1/2\})$ by 2 . By repeating this process we must eventually get the required result.

Now suppose that $f^{-1}(F \times \{1/2\})$ has components $F_1, \ldots, F_n$ with $n \geq 2$. We can still apply Stallings' method of arc-chasing, but the conclusion is slightly different. We start by choosing a base point e in $F \times \{1/2\}$ and corresponding base points e_i in F_i with $f(e_i) = e$. Now choose a path Γ' joining distinct components of $f^{-1}(F \times \{1/2\})$, chosen so that Γ' only meets F_i in e_i , for each i . As before, we can arrange that $f(\Gamma')$ is a null-homotopic loop in K based at e . We can also arrange that Γ' is transverse to each F_i . As before, we can obtain an equation $g_1 \ldots g_n = 1$ in $\pi_1(K) = A *_C B$ where the g_i's alternately lie in A and B . From this we deduce that some g_i lies in $C = \pi_1(F \times \{1/2\})$. By composing the path Γ_i joining F_1 to F_2 say with a suitable loop in F_1 we finally obtain a path Γ in M joining F_1 and F_2 and meeting no other components of $f^{-1}(F \times \{1/2\})$ such that $f(\Gamma)$ is a null-homotopic loop in K based at e .

Cut M along $f^{-1}(F \times \{1/2\})$ and let X be the component which contains Γ and of course has F_1 and F_2 in its boundary. Observe that X must be irreducible as M is irreducible and ∂X is incompressible in M . Also the existence of Γ implies that

$\pi_1(F_1, e_1) = \pi_1(F_2, e_1)$ if we use Γ to define the second group. I claim that this implies that $\partial X = F_1 \cup F_2$ and that the natural maps $\pi_1(F_i) \to \pi_1(X)$ are isomorphisms. We will prove this below. Assuming this for the moment, we know that X 'looks like' $F_1 \times I$. As the maps $\pi_1(F_i) \to \pi_1(X)$ are isomorphisms, we obtain an isomorphism $\pi_1(F_1) \to \pi_1(F_2)$. The corresponding map $F_1 \to F_2$ is homotopic to a homeomorphism by Theorem 3.3. Therefore we have two embeddings of F_1 in X with images F_1 and F_2 and by construction they are homotopic. This gives us a homotopy equivalence $F_1 \times I \to X$ which induces a homeomorphism of boundaries. Hence there is also a homotopy equivalence $X \to F_1 \times I$ which induces a homeomorphism of boundaries. Now we have two maps of F_1 to $F \times \{1/2\}$, one is f and the other comes from $f|F_2$ and our chosen heomeomorphism $F_1 \to F_2$. These two maps are homotopic as they induce the same isomorphism of fundamental groups. Hence they can be extended to a map $F_1 \times I \to F \times \{1/2\}$. We can now construct a map $g : M \to K$ so that g equals f outside X and $g(X) = F$, by using the homotopy equivalence $X \to F_1 \times I$. But as M and K are aspherical, g must be homotopic to f because they induce the same map of fundamental groups. By a small further homotopy of f we can remove X from $f^{-1}(F \times \{1/2\})$, and this achieves our goal of reducing by 2 the number of components of $f^{-1}(F \times \{1/2\})$. This completes the proof of the splitting theorem apart from the following result.

<u>Lemma 3.7</u>

Let X be a compact, orientable, irreducible 3-manifold and let F_1 and F_2 be two incompressible boundary components of X neither of which is S^2 . Let Γ be a path in X joining points e_i in F_i .

If $\pi_1(F_1,e_1) = \pi_1(F_2,e_1)$ when we define the second group using Γ, then $\partial X = F_1 \cup F_2$ and the natural maps $\pi_1(F_i) \to \pi_1(X)$ are ismorphisms.

Proof: Let $\tilde{X}$ be the covering space of X corresponding to $\pi_1(F_1,e_1) \subset \pi_1(X,e_1)$. Both F_1 and F_2 lift to $\tilde{X}$. Now consider the exact sequence $H_3(\tilde{X},\partial\tilde{X}) \to H_2(\partial\tilde{X}) \to H_2(\tilde{X})$. We know that $H_2(\tilde{X})$ has rank one, because $\tilde{X}$ is homotopy equivalent to F_1. We also know that $H_2(\partial\tilde{X})$ has rank at least two and $H_3(\tilde{X},\partial\tilde{X})$ has rank zero or one. It follows that $H_3(\tilde{X},\partial\tilde{X})$ has rank one and $H_2(\partial\tilde{X})$ has rank two. Hence $\tilde{X}$ is compact and $\partial\tilde{X} = F_1 \cup F_2$. The result follows as we must have $\tilde{X} = X$.

Chapter IV Waldhausen's results

Before stating and proving Waldhausen's results, we must discuss sufficiently large 3-manifolds. Let M be a compact, orientable, irreducible 3-manifold with non-empty boundary. If $H_1(M)$ is finite then ∂M consists of 2-spheres and so $M = D^3$. So we consider the case when $H_1(M)$ is infinite. Note that in this case M must be aspherical as $\pi_1(M)$ is infinite.

In the exact sequence $H^1(M) \xrightarrow{f} H^1(\partial M) \xrightarrow{g} H^2(M, \partial M)$, the homomorphisms f and g are dual maps. This can be seen from Poincaré duality in M and ∂M. If g were injective, then $\ker(g)$ would be trivial. Hence so would $\operatorname{coker}(f)$ be trivial. Thus f would be onto and so g would be zero. This would imply $H^1(\partial M) = 0$. Hence g must have kernel, and there is α in $H^1(M)$ with $f(\alpha)$ non-zero in $H^1(\partial M)$. Corresponding to α, we have a map $p : M \to S^1$, which we make transverse to a point v of S^1. By the usual argument, we can arrange $p^{-1}(v)$ is incompressible in M. Now the homology class in $H_1(\partial M)$ of $\partial(p^{-1}(v))$ is equal to the dual of $f(\alpha)$. As $f(\alpha) \neq 0$, there is a component F of $p^{-1}(v)$ such that the homology class of ∂F in $H_1(\partial M)$ is non-zero. In particular, F fails to separate M.

One can apply the same argument to M_1, which is M cut along F, to obtain a sequence of incompressible surfaces and 3-manifolds. In his paper [35] Waldhausen shows that if one chooses the surfaces with rather more care than we describe here, then the sequence terminates. Thus one obtains a sequence of 3-manifolds M_i and incompressible surfaces F_i in M_i such that M_{i+1} equals M_i cut along F_i, $M_o = M$ and M_n is the

3-ball for some n . We say that M <u>has a hierarchy</u>.

We say that a closed orientable, irreducible 3-manifold M is <u>sufficiently large</u> if M contains an incompressible closed surface F not S^2 . By cutting M along F , we obtain one or two irreducible 3-manifolds with boundary and so M itself has a hierarchy with first surface F . For convenience we say that any orientable, irreducible 3-manifold with non-trivial boundary is <u>sufficiently large</u>.

Examples of closed, orientable, irreducible 3-manifolds which are not sufficiently large do exist [4] but all the known examples have a finite covering space which is sufficiently large. It is not known if this must always be so. Observe that the second splitting theorem of Chapter III shows that M must be sufficiently large if $\pi_1(M)$ is an appropriate amalgamated free product. Also if $H_1(M)$ is infinite then M must be sufficiently large. This is because $H^1(M)$ is non-zero, so we can find an essential map $f : M \to S^1$. If we choose v in S^1 we can homotop f so that f is transverse to v and $f^{-1}(v)$ is incompressible in M . Now $f^{-1}(v)$ cannot be empty and all 2-sphere components can be removed . Thus some component of $f^{-1}(v)$ is an incompressible surface not S^2 .

Now we can state and prove the following results.

<u>Theorem 4.1</u> Suppose that M^3 , N^3 are compact, orientable and irreducible and N is sufficiently large . If $f : M \to N$ is a proper map, which induces an isomorphism of fundamental groups and induces a homeomorphism of ∂M into ∂N , then f is properly homotopic to a homeomorphism modulo ∂M .

Remark In fact, one must have $f(\partial M) = \partial N$ as ∂M bounds in M, but no proper subset of ∂N bounds in N.

Proof of Theorem 4.1 This is by induction on the minimal hierarchy length for N, say $\ell(N)$.

If $\ell(N) = 0$, then $N \cong D^3$. Hence $\pi_1(M)$ is trivial and $M \cong D^3$ also. The result follows.

To do the induction step, we have two cases.

Case 1 N is closed.

Observe that in this case, M must also be closed. Let F be the closed incompressible surface in N which starts the hierarchy for N. Then $\pi_1(N)$ is of the form $A *_C B$ or $A *_C$ where $C \cong \pi_1(F)$. Also N is aspherical. The methods used to prove Theorem 3.6 show that we can homotop f so that f maps $f^{-1}(F)$ homeomorphically to F. Thus if we cut N along F to get N_1 and M along $f^{-1}(F)$ to get M_1, we have $f : M_1 \to N_1$ which satisfies all the hypotheses of our theorem. Thus applying our induction hypothesis to the (one or two) components of M_1 and N_1 gives the required result.

Case 2 N has boundary

Let F be the incompressible surface in N which starts the hierarchy for N. We know that the homology class of ∂F in $H_1(\partial N)$ is non-zero so that in particular, F has boundary. We homotop $f: M \to N$ modulo ∂M to be transverse to F, and so that $f^{-1}(F)$ is incompressible in M. We also arrange that $f^{-1}(F)$ has no 2-sphere components. Let L be a component of $f^{-1}(F)$. The map $L \to F$ induced by f is proper and injects π_1 and is a homeomorphism of ∂L into ∂F. If follows that this map is homotopic to a covering map. But then this map must be a homeomorphism, as it induces a homeomorphism on ∂L. As L now contains all of $f^{-1}(\partial F)$, we see that $f^{-1}(F)$ must have only one component L mapping homeomorphically to F. As in Case 1, this allows us to use our induction hypothesis and prove the required result.

It is interesting to point out that in the case when M and N do have boundary, the above result does not use Stallings' arc-chasing method. The only results needed are the Loop Theorem and the Sphere Theorem and of course the existence of a hierarchy for N. We can now prove the following result. Many people have proved closely related results, the first being Stallings in [26].

Corollary 4.2 (the h-cobordism theorem)

If X^3 is irreducible, orientable and compact and $\pi_1(X)$ is isomorphic to $\pi_1(F)$ where F is a closed orientable surface not S^2, then $X \cong F \times I$.

Proof: X is homotopy equivalent to F, hence $H_3(X) = 0$. Therefore X is not closed. Also no component of ∂X can be a 2-sphere as this would imply $X \cong D^3$. Let L be a component of ∂X. If L were compressible in X, we would have a 2-disc D in X with boundary an essential circle in L. Now $\pi_1(X)$ cannot be a non-trivial free product, so D must separate X, and one component of $X - D$ must be a ball. But then ∂D must bound a 2-disc in L after all. We deduce that L must be incompressible in X.

Now Lemma 3.5 tells us that $\pi_1(L) = \pi_1(X)$, and, in particular, L is homeomorphic to F . As before, the exact sequence $H_3(X,\partial X) \to H_2(\partial X) \to H_2(X)$ shows that ∂X has exactly two components F_1 and F_2 . This tells us that there is a proper homotopy equivalence $F \times I \to X$ which induces a homeomorphism of boundaries. The corollary now follows from Theorem 4.1.

Armed with this Corollary, we can now extend Theorem 4.1.

Theorem 4.3 Suppose that M^3 , N^3 are compact, orientable and irreducible, N is sufficiently large, and ∂M and ∂N are incompressible in M and N respectively. If $f : M \to N$ is a proper map inducing an ismorphism of fundamental groups, then f is properly homotopic to a map g such that

(i) g is a homeomorphism,

or (ii) $g(M) \subset \partial N$. In this case M and N are homeomorphic to $F \times I$ for some closed surface F .

Proof: If ∂M is empty, the result follows by Theorem 4.1. Otherwise let L be a component of ∂M and F be the component of ∂N such that $f(L) \subset F$. As L and F are incompressible, we have $\pi_1(L) \subset f_*^{-1}(\pi_1(F)) \subset \pi_1(M)$ and we deduce $\pi_1(L) = f_*^{-1}(\pi_1(F))$. Hence we can homotop f so that for each component L of ∂M , f maps L homeomorphically onto some component of ∂N .

Suppose that two components L_1 and L_2 of ∂M have the same image. Choose a path Γ in M from L_1 to L_2 . By composing Γ with a suitable loop in M , as usual, we can arrange that $f(\Gamma)$ is a null-homotopic loop in N based at some point of ∂N. Hence, from Lemma 3.7 and Corollary 4.2, we see that $M \cong L_1 \times I$. Also Corollary 4.2 tells us that $N \cong L_1 \times I$. Clearly, we can homotop f into ∂N, to obtain case (ii) of our theorem.

Otherwise, we can suppose that f induces a homeomorphism of ∂M into ∂N. Thus we can apply Theorem 4.1 to obtain case (i) of our theorem.

Finally, we can extend this result to the following which is Waldhausen's general form of the result.

Theorem 4.4 Suppose that M^3, N^3 are compact, orientable and irreducible, N is sufficiently large, and ∂M and ∂N are incompressible in M and N respectively. If $f : M \to N$ is a proper map inducing an injection of fundamental groups, then f is properly homotopic to a map g such that

(i) g is a covering map,

or (ii) $g(M) \subset \partial N$. In this case M is homeomorphic to $F \times I$ for some closed surface F.

Proof: Let $\tilde{N}$ be the covering space of N corresponding to $f_*(\pi_1(M)) \subset \pi_1(N)$. Note that $\tilde{N}$ may not be compact. The map f lifts to $\tilde{f} : M \to \tilde{N}$ which is a proper map inducing an isomorphism of fundamental groups. Also $\partial\tilde{N}$ is incompressible in $\tilde{N}$.

If M is closed, then so must $\tilde{N}$ be. For $\tilde{f} : M \to \tilde{N}$ is a homotopy equivalence, so $H_3(\tilde{N}) \cong H_3(M)$. Therefore Theorem 4.1 applies to give us the required result.

Let L be a component of ∂M and F the component of $\partial\tilde{N}$ with $\tilde{f}(L) \subset F$. Then $\pi_1(L) \subset \tilde{f}_*^{-1}(\pi_1(F))$. It follows that F must be closed and that $\pi_1(L) = \tilde{f}_*^{-1}(\pi_1(F))$. Hence we can homotop $\tilde{f}$ so that for each component L of ∂M , f maps L homeomorphically onto some component of $\partial\tilde{N}$.

As before, if two components L_1 , L_2 of ∂M have the same image then $M \cong L_1 \times I$, and we can homotop $\tilde{f}$ into $\partial\tilde{N}$ obtaining case (ii) of our theorem. Otherwise we must have $\tilde{f}$ inducing a homeomorphism of ∂M onto $\partial\tilde{N}$. Again this is because ∂M bounds in M but no proper subset of $\partial\tilde{N}$ bounds in $\tilde{N}$. In particular $\partial\tilde{N}$ is compact and so must be a finite sheeted covering space of ∂N . Therefore $\tilde{N}$ is also a finite sheeted covering space of N and so $\tilde{N}$ must be compact. Theorem 4.1 applies to give case (i) of our theorem.

Note that in the above we assumed that $\tilde{N}$ was irreducible. This is correct as Waldhausen showed in [35] that the universal covering space of N is D^3 with some boundary removed which is irreducible, and any manifold covered by an irreducible manifold is also irreducible. However it is not known in general whether any covering space of an irreducible manifold is irreducible.

Chapter V The coherence of $\pi_1(M^3)$

We say that a group G is coherent if every finitely generated subgroup of G is finitely presented. In this chapter, I will present the proof that the fundamental group of any 3-manifold is coherent. This result was proved by myself and by Shalen independently using similar methods. One corollary of this result is that any Kleinian group is coherent. The result also has many applications to 3-manifold theory.

Some groups which are not coherent are easy to describe. We will give an example here. Let A be the free group on two generators a,b let B be the free group on two generators c,d, and let C be the free group on countably many generators. We embed C in A and B as the subgroups generated by $\{b^{-i}ab^{i}\}_{i\geq 1}$ and $\{d^{-i}cd^{i}\}_{i\geq 1}$ respectively. The group $G = A *_C B$ is clearly generated by a,b,c and d, but G is not finitely presented. This is because $H_2(G)$ is not finitely generated, which can be proved easily using the Mayer-Vietoris exact homology sequence of $A *_C B$. (This sequence is exactly the sequence associated to a corresponding union of $K(\pi,1)$'s where $\pi = A,B,C.$) It is also interesting to observe that G can be embedded in a finitely presented group H. Let ϕ be the automorphism of G defined by $\phi(a) = b^{-1}ab$, $\phi(b) = b$, $\phi(c) = d^{-1}cd$, $\phi(d) = d$. The corresponding extension H of G by Z has presentation

$$H = \{a,b,c,d,x: [x,b] = 1 = [x,d],\ x^{-1}ax = b^{-1}ab,\ x^{-1}cx = d^{-1}cd,\ b^{-1}ab = d^{-1}cd\},$$

and so is finitely presented.

The proof of the coherence result falls into two parts – group theory and geometry. We first do some group theory. The idea behind the result is the feeling that if one takes a free product $A * B$ and adds a relation, such as $a = b$ where $a \in A$, $b \in B$, the result is 'less free' then before. However, this feeling must be made much more precise. For the group $\{x,y:x^2=y^2\}$ has non-trivial centre and so is not a free product, but if we add the relation $x^2 = 1$ we obtain $Z_2 * Z_2$.

We define the <u>complexity</u> $c(G)$ of a finitely generated group to be the ordered pair $(r+s,s)$ where $G \cong G_1 * \ldots * G_r * F_s$ where each G_i is indecomposable and not isomorphic to Z. These pairs will be ordered lexicographically.

<u>Theorem 5.1</u> Let $H = H_1 * \ldots * H_{r+s}$ where each H_i is indecomposable and only $H_{r+1}, \ldots, H_{r+s}$ are isomorphic to Z. If $f : H \to G$ is an epimorphism such that f injects each of $H_1, \ldots H_r$ then f is an isomorphism or $c(G) < c(H)$.

<u>Remark</u> This result is a corollary of the main theorem of Higgins in [9], using the methods by which he deduces Grusko's theorem. However, we will present a topological proof using Stallings methods.

Proof: G is finitely generated and so $G = G_1 * \ldots * G_n$ where each G_i is indecomposable. As $f(H_i)$ is indecomposable, for $1 \leq i \leq r$, we hnow that $f(H_i)$ lies in a conjugate $g_i^{-1} G_j g_i$ of some G_j, by the Kuros Subgroup Theorem.

As in Stallings' proof of Grusko's theorem, we construct simplicial complexes X and Y whose fundamental groups are H and G respectively.

For each free factor H_i of H choose a complex X_i with fundamental group H_i and choose a vertex v_i of X_i. If $H_i \cong Z$, we choose X_i to be a circle. Attach a 1-simplex P_i to X_i by identifying one endpoint to v_i. We identify all the remaining endpoints of $P_1,\ldots,P_{r+s}$ to a single point a, the base point of X. We construct Y similarly as a union of $Y_1,\ldots,Y_n$ and 1-simplices $Q_1,\ldots Q_n$ and Y has basepoint b.

We can now choose a map $h:X\to Y$ inducing $f:\pi_1(X,a)\to\pi_1(Y,b)$ such that if $1\leq i\leq r$, then $h(X_i)$ is contained in some Y_j and P_i is mapped to a path consisting of a loop based at b representing g_i composed with the path Q_j. Further, by subdividing X and Y and performing homotopies of h we can arrange that $h^{-1}(b)$ consists of only finitely many points. This last condition is crucial, as it allows one to apply the method of arc chasing as in Stallings' proof of Grusko's theorem. Thus, by changing X, we can arrange that $h^{-1}(b)$ is a tree and hence H has a factorisation $K_1*\ldots*K_n$ such that $f(K_i) = G_i$.

Now we suppose that $c(G)\geq c(H)$ and prove that f is an isomorphism. As $n\geqq r+s$, the uniqueness of factorisation theorem tells us that $n=r+s$ and we can suppose $K_i\cong H_i$ with K_i conjugate to H_i if $1\leq i\leq r$. We know that f injects H_i if $1\leq i\leq r$, so f also injects K_i if $1\leq i\leq r$. Hence f maps K_i isomorphically to G_i. The remaining s factors $G_{r+1},\ldots,G_{r+s}$ are all cyclic as each G_i is a quotient of K_i and our assumption that $c(G)\geqq c(H)$ implies that each must be isomorphic to Z. Hence the restriction of f to each K_i is an isomorphism onto G_i and so f is an isomorphism. This completes the proof of the theorem.

For the purposes of the geometrical part of this result, we make the following definition. A submanifold N^3 of M^3 is <u>incompressible</u> if ∂N is incompressible in M. The geometrical method we use here was first used in [8] and [31] .

Theorem 5.2 If G is the fundamental group of a 3-manifold M , then G is coherent.

Proof: As any subgroup of G is also the fundamental group of a 3-manifold, it will suffice to prove that if G is finitely generated then G must be finitely presented. We shall prove this by induction on the minimal number of generators of G . The result is obvious if G is cyclic.

Suppose we have already proved our result for r-generator groups where $r < n$, and suppose G is a n-generator group. (We use 'n-generator' to mean that n is the minimal number of generators of G .) If G is decomposable, then each factor will have less than n generators and the result follows by induction. Therefore we suppose G is indecomposable.

Lemma 5.3 Let G be a n-generator indecomposable group such that every subgroup with less than n generators is finitely presented. Then there is an indecomposable finitely presented group H and an epimorphism $f : H \to G$ such that if H' is an intermediate quotient of H then H' is also indecomposable.

Proof: Let S be the set of all finitely presented n-generator groups A which admit a free factorisation $A = A_1 * \dots * A_r$ and an epimorphism $f : A \to G$ injecting each factor which is not isomorphic to Z . S is non-empty as it contains F_n . We choose a group A in S of minimal complexity. We will show that if $f : A \to G$ factors through a group B not equal to A then B is indecomposable. The lemma will then follow by taking H to be A with one relation added. (One can do this unless f is an isomorphism but then we will already have shown G to be finitely presented.)

Suppose that f factors through a group B which is isomorphic to a non-trivial free product $B_1 * B_2$. The images C_1 and C_2 of B_1 and B_2 in G each have less than n generators because B_1 and B_2 have. Hence C_1 and C_2 are both finitely presented and so B has a finitely presented quotient $C = C_1 * C_2$ through which f still factors. Now C has a factorisation obtained by factoring C_1 and C_2 and so every factor of C is injected into G . Hence C lies in the set S . Hence $c(C) \geq c(A)$ by the choice of A , and Theorem 5.1 implies that C = A and so B = A which is the required result.

Using the group H obtained here, we can now complete the proof of the coherence result. Let K be a finite simplicial complex with fundamental group H and let $\phi : K \to M$ be a piecewise linear map inducing the epimorphism $f : \pi_1(K) \to \pi_1(M)$. Let N be a regular neighborhood of $\phi(K)$ and let $A = \phi(\pi_1(K)) \subset \pi_1(N)$. Let $i : N \to M$ be the inclusion map. Then N is a compact submanifold of M which satisfies the following condition, because A is an intermediate quotient of H .

<u>Condition (*)</u> $\pi_1(N)$ contains an indecomposable subgroup A such that $i_*: A \to G$ is onto and any intermediate quotient of A is indecomposable.

If N is incompressible in M , then van Kampen's Theorem and Condition (*) imply that $\pi_1(N) = \pi_1(M)$. As N is compact, $\pi_1(N)$ is finitely presented which proves the required result.

If N is compressible in M, then Corollary 3.1 of the Loop Theorem gives us a 2-disc D embedded in M such that $D \cap \partial N = \partial D$ is an essential curve in ∂N. If D is not contained in N, we replace N by N' which is obtained from N by attaching a 2-handle whose core is D. N' also satisfies Condition (*), the appropriate subgroup of $\pi_1(N')$ being the image of A, under the natural map $\pi_1(N) \to \pi_1(N')$. If D is contained in N, we have two cases according as D separates N or not.

If D separates N, then $N = N_1 \cup N_2$ and $\pi_1(N) = \pi_1(N_1) * \pi_1(N_2)$. The subgroup A of $\pi_1(N)$ is indecomposable and not isomorphic to Z and so lies in a conjugate of $\pi_1(N_1)$ say. We replace N by N_1, the appropriate subgroup of $\pi_1(N_1)$ being the conjugate of A which lies in $\pi_1(N_1)$. If D fails to separate N, then $\pi_1(N) = \pi_1(N_1) * Z$ and we must have a conjugate of A lying in $\pi_1(N_1)$. Again we replace N by N_1.

We have shown how to replace N by a new manifold N' also satisfying Condition (*) if N is compressible. We repeat this process as long as possible. As in Corollary 3.2, $\Sigma(2-\chi_i)^2$ must decrease at each step where the summation is over all components of ∂N. Hence this process must terminate and we will then have obtained an incompressible submanifold of M satisfying Condition (*). This completes the proof of the coherence result.

To finish off this series of lectures, I would like to discuss the following question which is suggested by the proof of the coherence result. This proof shows that if $\pi_1(M)$ is finitely generated and indecomposable, then there is a compact submanifold N of M with the natural map $\pi_1(N) \to \pi_1(M)$ an isomorphism. In fact, this result is still true if we remove the restriction that $\pi_1(M)$ be indecomposable [25] . This raises the question of whether an open manifold M with finitely generated fundamental group is the interior of a compact 3-manifold or more generally whether M is obtained from a compact manifold N by removing a closed subset of ∂N . Such a manifold M is said to be almost compact. The answer to this question is negative, because of Whitehead's example of an open contractible 3-manifold not R^3 [38]. Tucker has given a characterisation of almost compact 3-manifolds [33].

This suggests that we ask whether M is almost compact if we also assume that the universal covering space is R^3 . The answer here is still negative, and the first counter example was given by Tucker [24]. We will give an example which is even simpler than Tucker's, though it is also less interesting.

In R^3 , take a plane Π and a disjoint path ℓ from some point P out to infinity which has infinitely many knots in it. See Fig. 1. The boundary Π' of a regular neighborhood of l is homeomorphic to R^2 . Let X be the closed region of R^3 between Π and Π' and construct M from X by identifying Π and Π' by a homeomorphism. Then $\pi_1(M) \cong Z$ by van Kampen's Theorem, and X has universal covering space R^3 . To see this second fact, we use the fact that R^3 is characterized by the property that every compact subset lies in the interior of a 3-ball. Finally let L be an embedded straight line path from P to Π . Then $\pi_1(X-L)$ is not finitely generated as it is an infinite amalgamated free product of non-trivial knot groups. Therefore $\pi_1(M-L)$ is also not finitely generated. Therefore M is not almost compact. For as everything is PL, $\pi_1(M-L)$ equals $\pi_1(M-U)$ where U is a regular neighborhood of L and if M were almost compact so would M - U be.

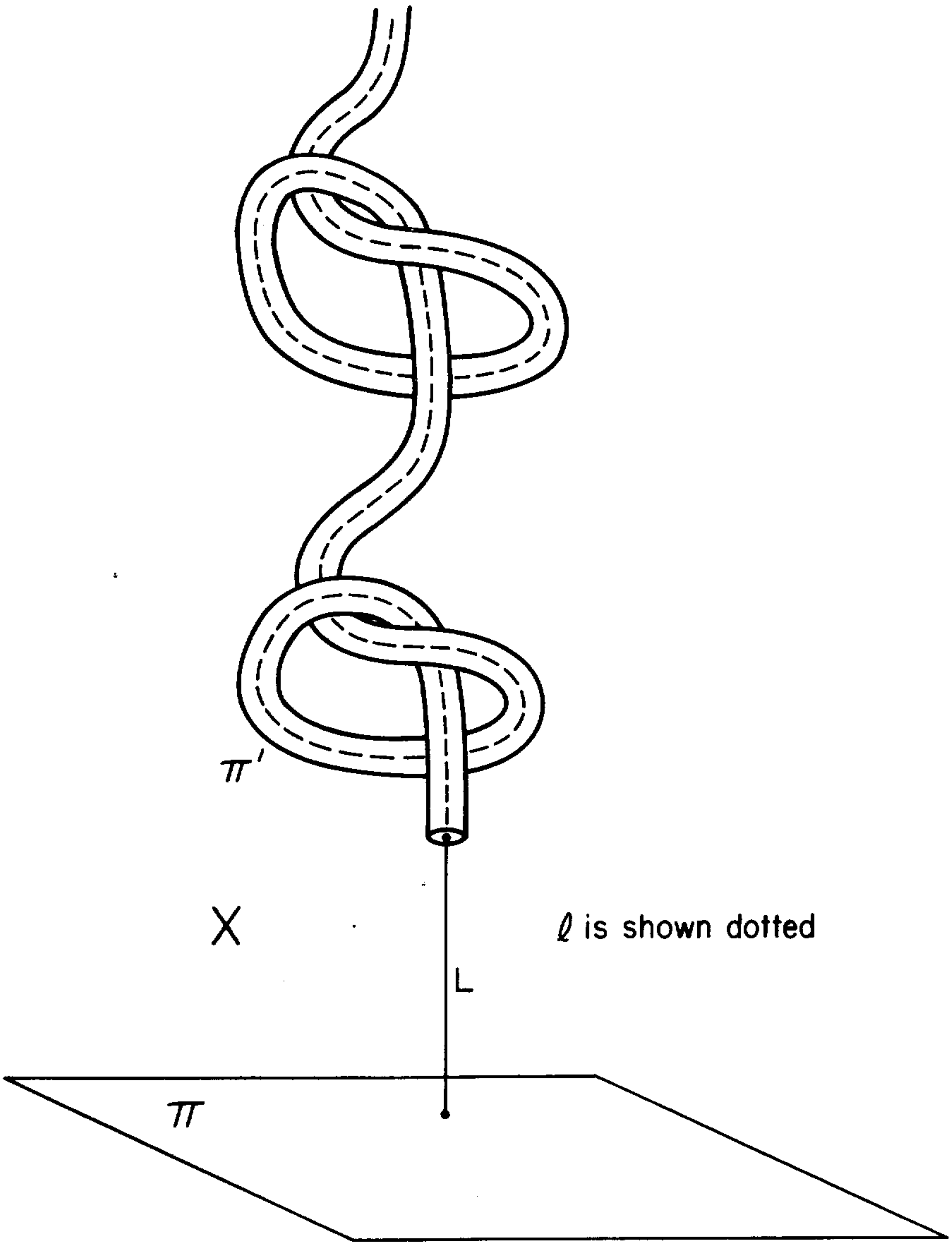

Fig. 1

It is interesting to note that the interior of X is homeomorphic to R^3 but X itself is not $R^2 \times I$. This is an example of two planes 'linking' in 3-space. Note that any two planes in 3-space are isotopic.

As a last ditch possibility, one asks whether M is almost compact assuming that the universal covering space of M is R^3_+ , but Tucker [34] has an example which disposes completely of this question also. Of course $\pi_1(M) = \pi_1(\partial M)$ is a surface group on this case. His initial example is of a 3-manifold N with universal covering space R^3_+ and $\pi_1(N) \cong Z$, but N is not almost compact. Note ∂N must be homeomorphic to $S^1 \times R$. Now choose any compact irreducible orientable 3-manifold M with non-empty boundary. Waldhausen has shown that such a manifold has universal covering space consisting of the 3-ball with some boundary removed [35]. Tucker then glues M to N by identifying closed annuli in the boundary of each which are incompressible in M and N respectively. The interior M_1 of the resulting manifold has universal covering space R^3 but is not almost compact.

Finally, carry out Tucker's construction with $M = F \times I$, gluing an annulus in $F \times \{1\}$ to an annulus in ∂N . Let M_1 be obtained by removing all the boundary of this union except for $F \times \{0\}$. Then the universal covering space of M_1 is R^3_+ and M_1 is not almost compact. Thus even the fundamental group of a closed orientable surface can 'act wildly' on R^3_+ .

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